



# Quantum gravity stability of isotropy in homogeneous cosmology

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## ABSTRACT

It has been shown that anisotropy of homogeneous spacetime described by the general Kasner metric can be damped by quantum fluctuations coming from perturbative quantum gravity in one-loop approximation. Also, a formal argument, not limited to one-loop approximation, is put forward in favor of stability of isotropy in the exactly isotropic case.

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## 1. Introduction

Standard [1] and loop [2–4] quantum cosmology heavily depends on the implicit assumption of (quantum) stability of general form of the metric. As a principal starting point in quantum cosmology, one usually chooses a metric of a particular (more or less symmetric) form. In the simplest, homogeneous and isotropic case, the metric chosen is the (flat) Friedmann–Lemaître–Robertson–Walker (FLRW) one. Consequently, (field theory) quantum gravity reduces to a much more tractable quantum mechanical system with a finite number of degrees of freedom. It is obvious that such an approach greatly simplifies quantum analysis of cosmological evolution, but under no circumstances is it obvious to what extent is such an approach reliable. The quantum cosmology approach could be considered unreliable when (for example) the assumed symmetry of the metric would be unstable due to quantum fluctuations. More precisely, in the context of the stability, one can put forward the two, to some extent complementary, issues (questions): (1) assuming a small anisotropy in the almost isotropic cosmological model, have quantum fluctuations a tendency to increase the anisotropy or, just the opposite, to reduce it? (2) assuming we start quantum evolution from an exactly isotropic metric should

be we sure that no quantum fluctuations are able to perturb the isotropy?

In this Letter, we are going to address the both issues of the quantum stability of spacetime metric in the framework of standard covariant quantum gravity. Namely, in Section 2, we address the first stability issue for an anisotropic (homogeneous) metric of the Kasner type, to one-loop in perturbative expansion. In Section 3, we give a simple, formal argument, not limited to one-loop, concerning the second issue.

## 2. One-loop stability

The approach applied in this section is a generalization of our approach used in [5] in the context of FLRW geometry. In our present work, the starting point is an anisotropic (homogeneous) metric,

$$ds^2 = dt^2 - a_1^2(t)(dx^1)^2 - a_2^2(t)(dx^2)^2 - a_3^2(t)(dx^3)^2, \quad (1)$$

of the Kasner type, i.e.

$$a_i^2(t) \equiv \left| \frac{t}{t_0} \right|^{2k_i}, \quad i = 1, 2, 3, \quad (2)$$

where  $k_i$  are the Kasner exponents. One should stress that we ignore any assumptions concerning matter content, and consequently, no prior bounds are imposed on  $k_i$ .

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In the perturbative approach

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \quad (3)$$

then

$$\kappa h_i(t) = 1 - \left| \frac{t}{t_0} \right|^{2k_i}, \quad h_i(t_0) = 0, \quad (4)$$

where  $\kappa = \sqrt{32\pi G_N}$ , with  $G_N$ —the Newton gravitational constant. The quantized field

$$h_{\mu\nu}(t, \mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_1(t) & 0 & 0 \\ 0 & 0 & h_2(t) & 0 \\ 0 & 0 & 0 & h_3(t) \end{pmatrix} \equiv \text{diag}(0, h_i(t)) \quad (5)$$

is small, as expected, closely to the expansion (reference) point  $t_0$ . Using the gauge freedom to satisfy the harmonic gauge condition (see, the second formula in (11)), we gauge transform the gravitational field  $h_{\mu\nu}$  as follows,

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (6)$$

where the gauge parameter

$$\xi_\mu(t) = \left( -\frac{1}{2} \int_0^t h(t') dt', 0, 0, 0 \right), \quad h(t) \equiv h_1(t) + h_2(t) + h_3(t). \quad (7)$$

Then,

$$h'_{\mu\nu}(t, \mathbf{x}) = \text{diag}(2\dot{\xi}_0(t), h_i(t)) = \text{diag}(-h(t), h_i(t)), \quad (8)$$

and, skipping the prime for simplicity, we have

$$h'_\lambda(t) = -2h(t), \quad (9)$$

where spacetime indices are being manipulated with the Minkowski metric  $\eta_{\mu\nu}$ . Now, we should switch from our present  $h_{\mu\nu}$  to standard perturbative gravitational variables, i.e. to the “barred” field  $\bar{h}_{\mu\nu}$  defined by

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h'_\lambda, \quad (10)$$

and

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \text{diag}(0, h_i(t) - h(t)) \quad \text{with } \partial^\mu \bar{h}_{\mu\nu} = 0. \quad (11)$$

The Fourier transform of  $\bar{h}_{\mu\nu}$  is

$$\tilde{\bar{h}}_{\mu\nu}(p) \equiv \tilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{p}) = (2\pi)^3 \delta^3(\mathbf{p}) \text{diag}(0, \tilde{h}_i(\omega) - \tilde{h}(\omega)), \quad (12)$$

where, for the  $h_i$  of the explicit form (4), we have (from now on, we denote classical gravitational fields with the superscript “c”)

$$\kappa \tilde{h}_i^c(\omega) = 2\pi \delta(\omega) + 2t_0^{-2k_i} \sin(\pi k_i) \Gamma(2k_i + 1) |\omega|^{-2k_i-1}. \quad (13)$$

According to (A.8) a one-loop quantum contribution corresponding to the classical metric (12) equals

$$\begin{aligned} \tilde{h}_{\mu\nu}^q(p) &= \frac{\pi \kappa^2 p^2}{2} \log\left(\frac{-p^2}{\Lambda^2}\right) \delta^3(\mathbf{p}) \\ &\quad \times [(2\alpha_1 \mathbb{E} + 4\alpha_2 \mathbb{P}) \text{diag}(0, \tilde{h}_i^c - \tilde{h}^c)]_{\mu\nu} \\ &= 2\pi \kappa^2 \omega^2 \log\left|\frac{\omega}{\Lambda}\right| \delta^3(\mathbf{p}) \text{diag}(\alpha_2 \tilde{h}^c, \alpha_1 \tilde{h}_i^c - (\alpha_1 + \alpha_2) \tilde{h}^c). \end{aligned} \quad (14)$$

Defining the auxiliary function

$$\tilde{h}_i^Q(\omega) \equiv \omega^2 \log\left|\frac{\omega}{\Lambda}\right| \tilde{h}_i^c(\omega), \quad (15)$$

we have

$$\begin{aligned} \kappa \tilde{h}_i^Q(\omega) &= 2\pi \delta(\omega) \omega^2 \log\left|\frac{\omega}{\Lambda}\right| \\ &\quad + 2t_0^{-2k_i} \sin(\pi k_i) \Gamma(2k_i + 1) |\omega|^{-2k_i+1} \log\left|\frac{\omega}{\Lambda}\right|. \end{aligned} \quad (16)$$

Its Fourier reverse is

$$\begin{aligned} \kappa h_i^Q(t) &= 2t_0^{-2k_i} k_i (2k_i - 1) |t|^{2k_i-2} \\ &\quad \times \left[ \psi(2 - 2k_i) + \frac{\pi}{2} \tan(\pi k_i) - \log|\Lambda t| \right], \end{aligned} \quad (17)$$

where  $\psi$  is the digamma function, and according to (14)

$$h_{\mu\nu}^q(t, \mathbf{x}) = \left( \frac{\kappa}{2\pi} \right)^2 \text{diag}(\alpha_2 h^Q(t), \alpha_1 h_i^Q(t) - (\alpha_1 + \alpha_2) h^Q(t)). \quad (18)$$

Performing the gauge transformation in the spirit of (7) we can remove the first (time) component in (18), and (once more, skipping the prime for simplicity) we get a quantum contribution to the Kasner metric

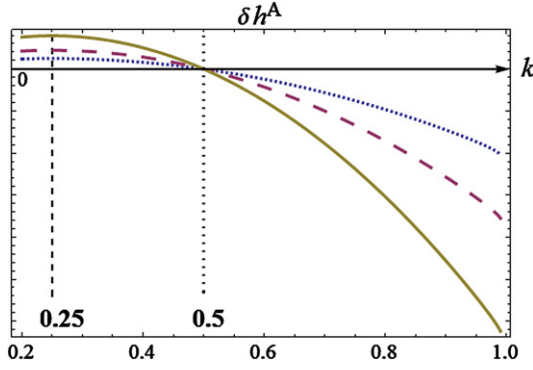
$$h_{\mu\nu}^q(t, \mathbf{x}) = \left( \frac{\kappa}{2\pi} \right)^2 \text{diag}(0, \alpha_1 h_i^Q(t) - (\alpha_1 + \alpha_2) h^Q(t)). \quad (19)$$

Only the “anisotropic” part of (19), i.e.

$$\delta h_i^A = \left( \frac{\kappa}{2\pi} \right)^2 \alpha_1 h_i^Q, \quad (20)$$

can influence the anisotropy of the evolution of the Universe. Since the dependence of  $\delta h_i^A$  on  $k_j$  is purely “diagonal” ( $\delta h_i^A$  depends only on  $k_j$  with  $j = i$ , see (17)), we have the following simple rule governing (de)stabilization of the isotropy: the increasing function  $\delta h^A(k)$  implies destabilization (there is a greater contribution of quantum origin to the metric in the direction of a greater classical expansion), whereas the decreasing function implies stabilization. Unfortunately,  $\delta h^A(k)$  is not a monotonic function because the digamma function  $\psi$  oscillates, and moreover (17) is (in general<sup>1</sup>) a  $\Lambda$ -cutoff dependent function. Nevertheless, if we assume the point of view that it is not necessary to expect or require the stability of the isotropy in the whole domain of the Kasner exponents  $k_i$ , but only for some subset of them, considered physically preferred, a definite answer emerges. Since  $k = \frac{1}{2}$  corresponds to radiation, and  $k = \frac{2}{3}$  corresponds to matter, we could be fully satisfied knowing that  $\delta h^A(k)$  is monotonic in the interval  $k \in (\frac{1}{4}, 1)$  ( $\supset (\frac{1}{2}, \frac{2}{3})$ ). Furthermore, since  $\alpha_1 > 0$  for any spin (see, Table A.1),  $\delta h^A(k)$  is a decreasing function in this interval, implying (quantum) damping of the anisotropy (see, Fig. 1).

<sup>1</sup> The quantum contribution is  $\Lambda$ -cutoff independent for  $k_i = \frac{1}{2}$  (a limit in (17) exists), i.e. for pure radiation (see, [5]). Intuitively, it could be explained by the fact that a scale-independent classical source, the photon field, implies vanishing of scale-dependent logarithms (no quantum “anomaly”).



**Fig. 1.** The drawing qualitatively presents 3 demonstration curves (for 3 different UV cutoffs  $\Lambda$ ) of the function  $\delta h^A(k)$  defined by (20) and (17). For the Kasner exponents  $k \in (\frac{1}{4}, 1)$ ,  $\delta h^A(k)$  is evidently decreasing function thus supporting the damping of cosmological anisotropy.

### 3. Above one-loop and final remarks

Section 2 has been limited to one-loop perturbative analysis of the stability of isotropy of cosmological evolution. But one can give a simple, formal argument ensuring stability of the “exactly isotropic” expansion, i.e. for

$$k_1 = k_2 = k_3, \quad (21)$$

which is perturbative but not limited to one-loop, making use of (A.1), where now  $D$  and  $\Pi$  is the full propagator and the full vacuum polarization, respectively. Since

$$h_{\mu\nu}^c = \text{diag}(0, h(t), h(t), h(t)), \quad (22)$$

no spatial coordinate  $x_i$  is singled out in (22), and consequently no spatial coordinate can be singled out on LHS of (A.1). This argument is only of purely formal interest as any other fluctuations can destabilize the isotropy.

Recapitulating, as far as perturbative quantum gravity in one-loop approximation is concerned we have observed that in a (hopefully) physically preferred region of the Kasner exponents  $k_i \in (\frac{1}{4}, 1)$  we should expect damping of anisotropy by quantum fluctuations, thus supporting reliability of the approach of quantum cosmology in this regime. One should point out that this result is subjected to several limitations. First of all, since  $\kappa h_i(t)$  should be small,  $t$  should be close to  $t_0$  because of (4). But for  $t \approx t_0$ , by virtue of (18), we have

$$h_{ii}^q \sim \kappa^2 t_0^{-2k} |t|^{2k-2} \sim \kappa^2 t_0^{-2} \sim \left( \frac{t_{\text{Planck}}}{t_0} \right)^2. \quad (23)$$

Therefore, to stay in the perturbative regime,  $t_0$  should be greater than  $t_{\text{Planck}}$ , and we have to be away from the (primordial) classical singularity. Instead, for  $t_0$  many orders greater than  $t_{\text{Planck}}$ , according to (23) the quantum contribution becomes small, and moreover classical matter (including radiation) is expected to begin to play a role. In particular, it was shown in [13]<sup>2</sup> that the effects of viscosity in the radiation and pressure from collisionless radiation ensure isotropization (and stability) of cosmological evolution at late times.

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**Table A.1**

The coefficients entering (A.8) taken from [8–11] (see, also [12]). In particular,  $\alpha_1$  enters (20), and its positivity for any spin supports the damping of cosmological anisotropy.

Spin	$\alpha_1$	$\alpha_2$	$\alpha_1 + \alpha_2$
0	$\frac{1}{480}$	$-\frac{1}{720}$	$\frac{1}{1440}$
$\frac{1}{2}$	$\frac{1}{160}$	$-\frac{1}{240}$	$\frac{1}{480}$
1	$\frac{1}{40}$	$-\frac{1}{60}$	$\frac{1}{120}$
2	$\frac{27}{80}$	$-\frac{59}{240}$	$\frac{11}{120}$

### Appendix A. One-loop vacuum polarization

For the reader's convenience, we present here a short derivation of the one-loop quantum correction to a classical gravitational field which is coming from (one-loop) vacuum polarization (self-energy) (for more details, see, [5], and also compare to [6,7] for  $\mathbf{x}$ -dependent but  $t$ -independent case).

In the momentum representation, the lowest order quantum corrections  $\tilde{h}_{\mu\nu}^q$  to the classical gravitational field  $\tilde{h}_{\mu\nu}^c$  are given by the formula

$$\tilde{h}_{\mu\nu}^q(p) = (D\Pi\tilde{h}^c)_{\mu\nu}(p), \quad (A.1)$$

where

$$D_{\mu\nu}^{\alpha\beta}(p) = \frac{i}{p^2} \mathbb{D}_{\mu\nu}^{\alpha\beta} \quad (A.2)$$

is the free graviton propagator in the harmonic gauge, and  $\Pi_{\mu\nu}^{\alpha\beta}(p)$  is the (one-loop) graviton vacuum polarization (self-energy) tensor operator. Here

$\mathbb{D} \equiv \mathbb{E} - 2\mathbb{P}$ , where

$$\mathbb{E}_{\mu\nu}^{\alpha\beta} \equiv \frac{1}{2}(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\nu^\alpha \delta_\mu^\beta), \quad \mathbb{P}_{\mu\nu}^{\alpha\beta} \equiv \frac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}. \quad (A.3)$$

Observing that

$$\bar{h}_{\mu\nu} = (\mathbb{D}h)_{\mu\nu}, \quad (A.4)$$

we get

$$\tilde{h}_{\mu\nu}^q(p) = \frac{i}{p^2} (\Pi\tilde{h}^c)_{\mu\nu}(p) = \frac{i}{p^2} (\Pi'\tilde{h}^c)_{\mu\nu}(p), \quad (A.5)$$

where the simplified by gauge symmetry version of  $\Pi$  equals

$$\Pi'(p) = \kappa^2 p^4 I(p^2) (2\alpha_1 \mathbb{E} + 4\alpha_2 \mathbb{P}), \quad (A.6)$$

with the coefficients  $\alpha_1$  and  $\alpha_2$  given in Table A.1, and

$$I(p^2) = -\frac{i}{(4\pi)^2} \log\left(-\frac{p^2}{\Lambda^2}\right) + \dots \quad (A.7)$$

Then the final formula assumes the form:

$$\tilde{h}_{\mu\nu}^q(p) = (i\kappa^2 p^2 I(p^2) (2\alpha_1 \mathbb{E} + 4\alpha_2 \mathbb{P}) \tilde{h}^c)_{\mu\nu}(p). \quad (A.8)$$

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<sup>2</sup> The author would like to thank the referee for pointing out the reference.

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